INTEGRAL GEOMETRY AND COMBINATORIAL ANALYSIS [1]

S. V. Koltsova, S. V. Polenkova

G. R. Derzhavin Tambov State University, Russia

The main aim of our work is the studying of combinatorial (or finite) Radon transform. Main notions have been formulated by Bolker in paper [1], where the Radon transform on k-element subsets of a finite set and on k-subspaces in a n-dimensional projective (or affine) space over a finite field was considered. Bolker found some sufficient conditions for the injectivity of this Radon transform and obtained some general inversion formulae (but without explicit expressions of coefficients) for above-mentioned cases.

In Section 1 of our paper we obtain necessary and sufficient conditions for the existence of an inversion formula for Radon transform on polyhedrons satisfying hypotheses of the Euler theorem. When an inversion formula exists, we present an algorithm that reconstructs a function defined on vertexes by means of sums of its values at the ends of each edge, i.e. by means of its Radon transform on edges.

In Section 2 we give explicit expressions for coefficients in Bolker's formula [1] for the Radon transform on k-element subsets of a n-element set (in [2] this formula was obtained in another way).

In Section 3 we generalize Bolker's formula to a Boolean.

In Section 4 we introduce a new variant of the Radon transform on a Boolean and obtain an inversion formula for this case.

We thank M.I. Graev and V.F. Molchanov for useful advices and discussions.

Let us introduce some notations and agreements.

For a finite set A, we denote by $|A|$ the number of elements of A and by $L(A)$ the linear space of complex valued functions on A. Introduce on $L(A)$ the standard inner product

$$(\varphi, \psi) = \sum_{a \in A} \varphi(a)\overline{\psi(a)}.$$

Then $L(A)$ is a Hilbert space (of dimension $|A|$).

Let B be a subset of A. Then by δ_B we denote the characteristic function of B. It is an element of $L(A)$. Note that the delta-function concentrated at a point $a \in A$ is the characteristic function of the one-element set $\{a\}$, i.e. $\delta_a = \delta_{\{a\}}$.

We use the following notation (we prefer $a^{[m]}$ to the Pochhammer symbol $(a)_m$) for generalized powers (shifted factorials):

$$a^{[m]} = a(a+1)\dots(a+m-1), \quad a^{(m)} = a(a-1)\dots(a-m+1).$$

1 Inversion formulae on polyhedrons

Let M be a regular (Platonic) polyhedron. Denote by V, E, F sets of vertexes, edges and faces of M respectively. Define a transform $R : L(V) \to L(E)$ as follows:

$$(Rf)(y) = \sum_{x \in y} f(x),$$

[1] Supported by the Russian Foundation for Basic Research (grant No. 05-01-00074a), the Scientific Programs "Universities of Russia" (grant No. ur.04.01.465) and "Devel. Sci. Potent. High. School", (Templan, No. 1.2.02).

where $x \in V$, $y \in E$. Let us call R the *Radon transform.*

The problem is to find an inversion formula for R, if possibly.

Theorem 1.1 *The inversion formula for Radon transform does exist for each regular polyhedron, except of the cube.*

The theorem can be generalized to the so-called Euler polyhedrons. We say that a polyhedron M is an *Euler polyhedron* if

$$|V| - |E| + |F| = 2.$$

An example is a polyhedron, homeomorphic to a sphere, every face of which is homeomorphic to a disc.

Theorem 1.2 *The inversion formula for Radon transform for Euler polyhedron does exist if and only if this polyhedron contains at least one face with odd number of edges.*

This theorem is also proved in [3].

2 Inversion formulae on finite sets. The first variant

Let X be a set with n elements, Y a family of subsets of X. Define a Radon transform $R : L(X) \to L(Y)$ by

$$(Rf)(y) = \sum_{x \in y} f(x). \tag{1}$$

In particular, we have

$$(R\delta_A)(y) = |A \cap y| \tag{2}$$

For $x \in X$, denote by $G(x)$ the set of elements $y \in Y$ containing this element x. Then by (2) we have

$$R\delta_x = \delta_{G(x)}. \tag{3}$$

Let $\{\delta_x\}$ and $\{\delta_y\}$ be the standard bases consisting of δ-functions in $L(X)$ and $L(Y)$, respectively. The matrix of the Radon transform in these bases coincides with the $(0,1)$-incidence matrix of X with Y.

We put on Y the following condition of regularity [1]. Let

1) the number of elements in $G(x)$ does not depend on x: $|G(x)| = \alpha$ for all $x \in X$;

2) $|G(x) \cap G(x')| = \beta \neq \alpha$ for all $x \neq x'$.

Let us call these conditions 1) and 2) the *B-conditions.* These conditions are standard for block designs in combinatorics.

If the B-conditions are satisfied then the Radon transform is injective [1] (it is the same that the matrix of R has rank $|X|$).

Let R^* be the adjoint transform to the Radon transform R, it is a map $L(Y) \to L(X)$ such that

$$(Rf, \varphi) = (f, R^*\varphi),$$

where inner products are taken in $L(X)$ and $L(Y)$ respectively. It is given by

$$(R^*\varphi)(x) = \sum_{y \in G(x)} \varphi(y). \tag{4}$$

Let Y be the set of *all* k-element subsets of X. The number of elements in Y is equal to $\binom{n}{k}$. In this case the B-conditions are satisfied, with

$$\alpha = \binom{n-1}{k-1}, \quad \beta = \binom{n-2}{k-2},$$

so that the Radon transform R is invertible. Applying the transform R^* to (3), we obtain

$$(R^*R\delta_x) = \left(\binom{n-1}{k-1} - \binom{n-2}{k-2}\right)\delta_x + \binom{n-2}{k-2}\delta_X = \binom{n-2}{k-1}\delta_x + \binom{n-2}{k-2}\delta_X, \tag{5}$$

whence

$$\delta_x = \binom{n-2}{k-1}^{-1} R^*R\delta_x - \frac{k-1}{n-k}\delta_X. \tag{6}$$

Now it is easy to obtain the inversion formula for R: expand an arbitrary function $f \in L(X)$ with respect to the basis $\{\delta_x\}$:

$$f = \sum_{x \in X} f(x)\delta_x. \tag{7}$$

and substitute (6) in (7), we obtain

$$f = \binom{n-2}{k-1}^{-1} R^*Rf - \frac{k-1}{n-k}\omega(f), \tag{8}$$

where

$$\omega(f) = \sum_{x \in X} f(x)$$

is the *total weight* of the function f. Let us express $\omega(f)$ in terms of Rf. Similarly to (2) we have

$$(R^*\delta_B)(x) = |B \cap x|, \quad B \subset Y.$$

Setting here $B = Y$, we obtain

$$(R^*\delta_Y)(x) = \binom{n-1}{k-1}$$

or

$$R^*\delta_Y = \binom{n-1}{k-1}\delta_X$$

Therefore

$$\omega(f) = \sum_{x \in X} f(x) = \binom{n-1}{k-1}^{-1} \omega^*(Rf), \tag{10}$$

where ω^* is the total weight of the function Rf, i.e.

$$\omega^*(Rf) = \sum_{y \in Y} (Rf)(y).$$

Substituting (10) in (8) we get

$$f = \binom{n-2}{k-1}^{-1} \left(R^*Rf - \frac{k-1}{n-1}\omega^*(Rf)\right). \tag{11}$$

It can be rewritten in the operator form:

$$E = \binom{n-2}{k-1}^{-1} \left(R^* - \frac{k-1}{n-1} \cdot I^*\right) R,$$

where E is the identity operator, I^* the operator $L(Y) \to L(X)$ whose matrix consists only from units.

In our paper [3] formula (11) is obtained in another way.

3 Inversion formulae on finite sets. The second variant

Let X be a set with n elements (points) again. Denote by Y^k the set of all k-element subsets of X, $k = 0, 1, 2, \dots, n$ (so that $Y^0 = \emptyset, Y^n = \{X\}$). Define a transform (an "integration") $R_{lk} : L(Y^k) \longrightarrow L(Y^l)$, $k \leqslant l$, by

$$(R_{lk}f)(y) = \sum_{x \subset y} f(x), \quad x \in Y^k, \ y \in Y^l.$$

Notice that R_{kk} is the identity operator E on $L(Y^k)$.

If $k + l > n$, then $|Y^k| > |Y^l|$, so that R_{lk} has a nontrivial kernel, hence an inversion formula for R_{lk} does not exist. So we consider $k + l \leqslant n$. Then R_{lk} is injective. The principal case is $k + l = n$. Then $|Y^k| = |Y^l|$, and R_{lk} is an isomorphism. To this case we can reduce the case $k + l < n$, here the inversion formula can be written in different forms.

Theorem 3.1 *Let $1 \leqslant k < l$, $k + l = n$. The following inversion formula for R_{lk} takes place:*

$$f(x) = \binom{l}{l-k}^{-1} \sum_{m=0}^{k} (-1)^m \frac{(l-k)^{[m]}}{k^{(m)}} \sum_{|x \cap y| = k-m} (R_{lk}f)(y) \tag{12}$$

This theorem can be formulated in another way. Define an operator $R_{kl}^{(p)} : L(Y^l) \longrightarrow L(Y^k)$ by

$$(R_{kl}^{(p)}\varphi)(x) = \sum_{|x \cap y| = p} \varphi(y).$$

Then for $1 \leqslant k < l$, $k + l = n$, there is the following decomposition of the operator E:

$$E = \binom{l}{l-k}^{-1} \sum_{m=0}^{k} \frac{(l-k)^{[m]}}{(-k)^{[m]}} R_{kl}^{(k-m)} R_{lk}.$$

The proof of the theorem is given in our paper [2].

4 Inversion formulae on finite sets. The third variant

Here we write a formula that recovers a function on Y^k by its "integrals" $(R_{lk}f)(y)$ over $y \in Y^l$ using all $l > k$.

The adjoint operator to R_{lk} is an operator $R_{kl} : L(Y^l) \longrightarrow L(Y^k)$ defined by

$$(R_{kl}\varphi)(x) = \sum_{x \subset y} \varphi(y), \quad x \in Y^k, \ y \in Y^l.$$

Theorem 4.1 *Let $1 \leqslant k < n/2$. Each function $f \in L(Y^k)$ is recovered by its integrals $R_{lk}f$, $l > k$, in the following way:*

$$f(x) = \sum_{j=1}^{n-k} (-1)^{j+1} (R_{k,k+j} R_{k+j,k} f)(x),$$

In the operator form this formula can be written as:

$$E = \sum_{j=1}^{n-k} (-1)^{j+1} R_{k,k+j} R_{k+j,k}. \tag{13}$$

Proof. Let $l \geqslant k$. Let us compute the composition $A_{kl} = R_{kl}R_{lk}$. We have

$$(A_{kl}f)(x) = \sum_{x\subset y}\sum_{y\supset u} f(u),$$

where $x, u \in Y^k, y \in Y^l$. Invert the order of summations, then

$$(A_{kl}f)(x) = \sum_{u} f(u) \sum_{y\supset(x\cup u)} 1, \tag{14}$$

Let $|x \cap u| = k - s, s = 0, 1, \ldots, k$. Then the complements of the set $x \cup u$ in X and in y consist of $n - k - s$ and $l - k - s$ points respectively. Therefore, the inner sum in (14) is equal to the amount of ways by which we can take $l - k - s$ points from $n - k - s$ ones. Hence

$$(A_{kl}f)(x) = \sum_{s=0}^{k} \binom{n-k-s}{l-k-s} \sum_{|x\cap u|=k-s} f(u).$$

Let us take the alternated sum of A_{kl} over l:

$$\sum_{l=k}^{n}(-1)^{l-k}(A_{kl}f)(x) = \sum_{s=0}^{k}(-1)^s \sum_{l=k+s}^{n}(-1)^{l-k-s}\binom{n-k-s}{l-k-s} \sum_{|x\cap u|=k-s} f(u). \tag{15}$$

This sum is equal to zero, since the alternated sum of numbers in a line of the Pascal triangle is zero if the line consists of more than one numbers. The latter is provided by the condition $k < n/2$. So we have

$$\sum_{l=k}^{n}(-1)^{l-k}A_{kl} = 0, \tag{16}$$

which is just (13) because $A_{kk} = E$. □

Remark. Let $k \geqslant n/2$. Then there exists $s = 0, 1, \ldots, k$ such that $s = n - k$. For this s the sum over l in (15) consists of only one term equal to 1. Instead of (16) we obtain

$$\sum_{l=k}^{n}(-1)^{l-k}(A_{kl}f)(x) = (-1)^{n-k} \sum_{|x\cap u|=2k-n} f(u).$$

Therefore, instead of (13) we get

$$E = \sum_{j=1}^{n-k}(-1)^{j+1}R_{k,k+j}R_{k+j,k} + (-1)^{n-k}R_{kk}^{(2k-n)}.$$

REFERENCES

1. E. T. Bolker. The finite Radon transform. Contemp. Math., 1987, vol. 63, 27–49.

2. S. V. Koltsova, S. V. Polenkova. Elements of finite integral geometry. Vestnik RUDN, ser. Matem., 2004, tom 11, No. 1, 54–63.

3. S. V. Koltsova, S. V. Polenkova. The Radon problem on polyhedrons. Vestnik Tambov Univ., 2005, tom 10, issue 3, 319–323.